

SYMPLECTIC EMBEDDINGS OF 4-DIMENSIONAL ELLIPSOIDS: ERRATUM

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The paper [4] gives necessary and sufficient conditions for one 4-dimensional ellipsoid to embed symplectically in another. Emmanuel Opshtein pointed out that one step in the proof of sufficiency (Theorem 3.11) was unjustified. When embedding an ellipsoid into a ball, the idea is to embed a small ellipsoid into the ball, perform a blow up procedure to convert this ellipsoid into two chains of embedded spheres \mathcal{S} , and then “inflate” normal to these spheres, a process that has the effect of deforming the symplectic form ω_0 so as to increase the relative size of the configuration \mathcal{S} and hence of the ellipsoid. The inflation process described in Claim 2 of the proof of [4, Theorem 3.11] used an embedded J -holomorphic curve C_A in an appropriate homology class A that intersects all the spheres in \mathcal{S} ω_0 -orthogonally. However the existence of the embedded curve C_A was not properly established. Although an embedded representative of A must exist for a generic ω -tame almost complex structure J , in the ellipsoidal context one must work with a *non-generic* J , one for which all the embedded spheres in \mathcal{S} are also holomorphic. At the time of writing, it is still unclear whether or not such an embedded curve must always exist, a question discussed further in [6]. This question is resolved in the affirmative when all spheres in \mathcal{S} have zero first Chern class: cf. Biran [1, Lemma 2.2B]. However, this condition is usually not satisfied in our situation.

There are three places where this problem affects the argument:

- (I) in the proof of [4, Theorem 3.11];
- (II) in the proof of [4, Corollary 1.6(i)] that claims spaces of embeddings of ellipsoids into ellipsoids are path connected; and
- (III) in the application of these ideas in [5, Lemma 2.18(i)].

In Case (I), it turns out to be easy to fill the gap using the more general inflation procedure developed by Li–Usher in [3]. We assume below that (M, ω_0) is a closed symplectic 4-manifold that contains a collection \mathcal{S} of symplectically embedded and ω_0 -orthogonally intersecting surfaces C_i^S , $1 \leq i \leq L$, in classes S_i and of self-intersections $S_i \cdot S_i = k_i$. A component C_i^S is called positive (resp. negative) if $k_i \geq 0$ (resp. $k_i < 0$).

Date: May 1, 2013.

2000 Mathematics Subject Classification. 53D35.

Key words and phrases. symplectic embedding, pairwise connect sum, symplectic inflation, partially supported by NSF grant DMS 0905191.

Lemma 1.1. *With \mathcal{S} , let $m_i \geq 0$ be integers such that the class $A := \sum_{i=1}^L m_i S_i$ satisfies $A \cdot S_i \geq 0$ for all i . Then for each $\kappa > 0$ there is a symplectic form ω_κ on M such that*

- (i) $[\omega_\kappa] = [\omega_0] + \kappa \text{PD}(A)$, where $\text{PD}(A)$ denotes the Poincaré dual of A ;
- (ii) *the restriction of ω_κ to each smooth component C_i^S of \mathcal{S} is symplectic.*

Proof. First note that, given any symplectic form ω satisfying (ii), one can use Gompf's pairwise sum construction as in [3, Theorem 2.3] to construct a new symplectic form ω'_i that still satisfies (ii) and lies in class $[\omega] + \varepsilon \text{PD}(S_i)$ provided only that $\omega(S_i) + \varepsilon k_i = \int_{S_i} [\omega] + \varepsilon \text{PD}(S_i) > 0$. Thus for negative components C_i^S one must choose $\varepsilon < \frac{\omega(S_i)}{|k_i|}$, while ε can be arbitrary for positive components. Notice also that $\omega'_i(S_j) \geq \omega(S_j)$ for all $i \neq j$. In other words, inflating by ε along C_i^S increases the size of all the curves in \mathcal{S} except perhaps for C_i^S itself.

Now choose ε_0 so that

$$0 < \varepsilon_0 < \min_{1 \leq i \leq L, m_i > 0, k_i < 0} \frac{\omega_0(S_i)}{m_i |k_i|}.$$

Then, the class $[\omega_0] + \varepsilon_0 m_i \text{PD}(S_i)$ evaluates positively on each $S_j, j = 1, \dots, L$. Hence, for each $\kappa \leq \varepsilon_0$ we can inflate by the amount $m_i \kappa$ along each curve $C_i^S, i = 1, \dots, L$, in turn, to construct a form ω_κ satisfying (i) and also (ii).

We now repeat this process starting with ω_{ε_0} instead of ω_0 . Note that, although ω_{ε_0} is nondegenerate on each C_i^S , the area has been redistributed, concentrating near the points where C_i^S intersects the other components C_j^S . However, because $A \cdot S_i \geq 0$, the total area of C_i^S does not decrease, i.e. $\omega_{\varepsilon_0}(S_i) \geq \omega_0(S_i)$ for all i . Hence the next step can be of size $\varepsilon_1 \geq \varepsilon_0$, where

$$\varepsilon_0 \leq \varepsilon_1 < \min_{1 \leq i \leq L, m_i > 0, k_i < 0} \frac{\omega_{\varepsilon_0}(S_i)}{m_i |k_i|}.$$

We therefore may construct a suitable form ω_κ for any given κ by a finite number of such steps. \square

In the application, when one is embedding one ellipsoid into another, one starts with a singular set \mathcal{S}' in a blow up of $\mathbb{C}P^2$ that consists of four chains of spheres, two from the inner approximation to the larger ellipsoid and two from the outer approximation to the smaller ellipsoid. We need to inflate along an integral class $A' := qA$ with $(A')^2 > 0$ as described in [4, Theorem 3.11]. Here A is chosen so that

- $A \cdot S_i = 0$ for all C_i^S in \mathcal{S}' ,
- $A \cdot E \geq 0, E \in \mathcal{E}$, where \mathcal{E} is the set of exceptional classes, i.e. classes that can be represented by symplectically embedded spheres of self-intersection -1 .

Given any ω -tame J such that the curves in \mathcal{S}' have J -holomorphic representatives, Taubes–Seiberg–Witten theory implies that A' has some connected J -holomorphic nodal representative $\Sigma^{A'}$. If we choose a generic ω -tame J with holomorphic restriction to \mathcal{S}' , then all the components of $\Sigma^{A'}$ that do not coincide with a component of \mathcal{S}' must be multiple covers of curves that are regular in the Gromov–Witten sense. Hence if

we also assume that J is integrable near \mathcal{S}' , then by standard arguments as described in [3] (with small modifications to keep \mathcal{S}' holomorphic as in [6]) we may alter J away from \mathcal{S}' and perturb the nodal curve to obtain a J' -holomorphic nodal representative $\Sigma^{A'}$ of A' such that

- each smooth component of $\Sigma^{A'}$ is a multiple cover of an embedded curve that either is a curve in \mathcal{S}' , or is an exceptional sphere, or has nonnegative self-intersection; in all cases it has nonnegative intersection with A' ;
- all intersections of these curves with each other and with \mathcal{S}' are ω -orthogonal.

Hence, if we define \mathcal{S} to consist of all the spheres in \mathcal{S}' together with the embedded curves underlying the components of $\Sigma^{A'}$, the class A' has a representative of the form considered in Lemma 1.1. Therefore we can fill the gap in Case (I) by using this lemma.

In Case (II), one must check that this argument applies when the exceptional set \mathcal{S} and class A' are fixed, but one starts with a family of symplectic forms $\omega_t, t \in [0, 1]$, each of which is nondegenerate on the spheres in \mathcal{S} . For this it suffices to show that a generic choice of ω_t -tame J_t , holomorphic near \mathcal{S} , can be perturbed on $M \setminus \mathcal{S}$ so that there is a suitable 1-parameter family of J'_t -holomorphic nodal curves $\Sigma_t^{A'}$ whose topological type does not vary with t . Since bubbling for closed curves occurs in codimension ≥ 2 , this is always possible. For more details, see [6].

Finally in Case (III) we need some additional facts about the representation of the exceptional classes $E \in \mathcal{E}$. These are not hard to establish by the methods described above: cf. [6]. However, in the case considered in [5] all the spheres in \mathcal{S}_E have self-intersection -2 and hence vanishing first Chern class. In this case, we can appeal to Biran [1, Lemma 2.2B], which shows that the requisite embedded representatives of the E -curves always exist.

Acknowledgement. I wish to thank Emmanuel Opshtein very warmly for pointing out the gap in [4], and for working with me on the construction of embedded curves.

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